Kinetic Equations

Text of the Exercises

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Exercise 1

Let f be a measurable function such that $d_f(\gamma) < +\infty$ for any $\gamma > 0$. Let $M > 0$. Define $f_M^-(x) := \chi_{\{|f| \leqslant M\}}(x) f(x)$ and $f_M^+ := f - f_M^-$. Prove that

$$d_{f_M^-}(\gamma) := \begin{cases} d_f(\gamma) - d_f(M), & \text{if } \gamma < M, \\ 0, & \text{if } \gamma \geqslant M, \end{cases} \tag{1}$$

$$d_{f_M^+}(\gamma) := \begin{cases} d_f(M), & \text{if } \gamma \leqslant M, \\ d_f(\gamma), & \text{if } \gamma > M. \end{cases} \tag{2}$$

Exercise 2

Prove the weak Young inequality for $p = 1$, i.e., that for any $q \in (1, +\infty)$ there exists a constant $C_q > 0$ such that for any $f \in L^1(\mathbb{R}^d)$, $g \in L^{q,\infty}(\mathbb{R}^d)$

$$\|f * g\|_{L^{q,\infty}(\mathbb{R}^d)} \leqslant C_q \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^{q,\infty}(\mathbb{R}^d)}. \tag{3}$$

Exercise 3

Let $[a, b] \subset \mathbb{R}$ be a compact interval. Assume that $f \in C^1([a, b])$ such that $f(x) > 0$ for any $x \in [a, b]$. Suppose that there exists $C > 0$ such that $f' \in C([a, b])$ satisfies

$$f'(x) \leqslant C f(x) [1 + |\log(f(x))|], \qquad \forall x \in [a, b]. \tag{4}$$

Prove that this implies that

$$f(x) \leqslant \exp((1 + |f(a)|) \exp(C(x - a))), \qquad \forall x \in [a, b]. \tag{5}$$